Fourier Analysis

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Review:

Prop 1. Let $f \in M(\mathbb{R})$. Then the following hold:

① $f(x+h) \overset{\mathcal{F}}{\longmapsto} \hat{f}(\xi) \cdot e^{2\pi i \xi h}, \quad \forall h \in \mathbb{R}.$

② $f(x) \cdot e^{-2\pi i h x} \overset{\mathcal{F}}{\longmapsto} \hat{f}(\xi + h), \quad \forall h \in \mathbb{R}.$

③ Let $\delta > 0$. Then

$$f(\delta x) \overset{\mathcal{F}}{\longmapsto} \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right).$$

④ $f'(x) \overset{\mathcal{F}}{\longmapsto} \hat{f}(\xi) \cdot (2\pi i \xi), \quad f' \in M(\mathbb{R}).$

④' $f^{(n)}(x) \overset{\mathcal{F}}{\longmapsto} \hat{f}(\xi) \cdot (2\pi i \xi)^n, \quad$ if $f^{(j)} \in M(\mathbb{R})$, $1 \leq j \leq n$.

⑤ $-2\pi i x f(x) \overset{\mathcal{F}}{\longmapsto} \dfrac{d \hat{f}(\xi)}{d\xi}, \quad x f(x) \in M(\mathbb{R}).$

⑤' $(-2\pi i x)^n f(x) \overset{\mathcal{F}}{\longmapsto} \dfrac{d^n \hat{f}(\xi)}{d\xi^n}, \quad x^n f(x) \in M(\mathbb{R}).$

Here we prove (5). We need to show that when $xf(x) \in M(\mathbb{R})$,

$$(*) \quad \lim_{\varepsilon \to 0} \frac{\hat{f}(\xi+\varepsilon) - \hat{f}(\xi)}{\varepsilon} = \int_{\mathbb{R}} f(x) \cdot (-2\pi i x) e^{-2\pi i \xi x} \, dx$$

for all $\xi \in \mathbb{R}$.

Notice that the (LHS) of $(*)$ is equal to

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{f(x)\left(e^{-2\pi i(\xi+\varepsilon)x} - e^{-2\pi i \xi x}\right)}{\varepsilon} \, dx$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \cdot \frac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon} \, dx.$$

Set $g_\varepsilon(x) = f(x) e^{-2\pi i \xi x} \cdot \dfrac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon}$, $x \in \mathbb{R}$.

Observe that

$$\left| \frac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon} \right| = \left| \frac{2 \sin(\pi \varepsilon x)}{\varepsilon} \right| \le 2\pi |x|$$

(using $|\sin y| \le |y|$),

Hence $$|g_\varepsilon(x)| \leq 2\pi|x||f(x)|.$$

Also notice that $$\lim_{\varepsilon\to 0} g_\varepsilon(x) = f(x)\, e^{-2\pi i \xi x}\cdot(-2\pi i x).$$

Applying the dominated convergence theorm (see below for the statement), we obtain

$$\lim_{\varepsilon\to 0} \frac{\hat{f}(\xi+\varepsilon)-\hat{f}(\xi)}{\varepsilon} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} g_\varepsilon(x)\,dx$$
$$= \int_{\mathbb{R}} \lim_{\varepsilon\to 0} g_\varepsilon(x)\,dx$$
$$= \int_{\mathbb{R}} f(x)\, e^{-2\pi i\xi x}(-2\pi i x)\,dx.$$

Hence

$$\frac{d\hat{f}(\xi)}{d\xi} = \int_{\mathbb{R}} f(x)\cdot(-2\pi i x)\, e^{-2\pi i \xi x}\,dx.$$

$\boxtimes$

Here is a version of the dominated convergence Thm in real analysis.

Thm (Lebesgue's dominated convergence Thm).

Let $(g_t)_{t\in(a,b)} \subset \mathcal{M}(\mathbb{R})$. Assume that $\exists\, h \in \mathcal{M}(\mathbb{R})$ such that

$$|g_t(x)| \leq h(x) \quad \text{for all } x\in\mathbb{R} \text{ and } t\in(a,b).$$

Suppose furthermore that

$$\lim_{t\to t_0} g_t(x) = f(x), \quad \forall\, x\in\mathbb{R}.$$

Then

$$\lim_{t\to t_0} \int_{\mathbb{R}} g_t(x)\,dx = \int_{\mathbb{R}} f(x)\,dx.$$

§ 5.3 An important example.

Example 1: Let $f(x) = e^{-\pi x^2}$. Show that $\hat{f}(\xi) = e^{-\pi \xi^2}$.

Pf.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i \xi x} \, dx$$

Taking derivative (using ⑤ of Prop 1),

$$\frac{d\hat{f}(\xi)}{d\xi} = \int_{-\infty}^{\infty} (-2\pi i x) f(x) e^{-2\pi i \xi x} \, dx$$

(check $x f(x) \in M(\mathbb{R})$)

$$= \int_{-\infty}^{\infty} (-2\pi i x) e^{-\pi x^2} e^{-2\pi i \xi x} \, dx$$

$$= \int_{-\infty}^{\infty} i \left(e^{-\pi x^2}\right)' \cdot e^{-2\pi i \xi x} \, dx$$

By Prop 1 (4)

$$= i \, (2\pi i \xi) \, \hat{f}(\xi)$$

$$= -2\pi \xi \, \hat{f}(\xi).$$

That is, $\hat{f}'(\xi) = -2\pi\xi \, \hat{f}(\xi)$.

Define $\quad G(\xi) = \hat{f}(\xi)\, e^{\pi \xi^2}.$

Taking derivative gives that

$$G'(\xi) = \left(\hat{f}(\xi)' + 2\pi\xi\, \hat{f}(\xi) \right) \cdot e^{\pi\xi^2}$$
$$= 0.$$

Hence $\quad G(\xi) = \text{constant}$
$$= \hat{f}(0).$$

But $\quad \hat{f}(0) = \int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} e^{-\pi x^2} dx.$

To estimate this integration, notice that

$$\left(\int_{\mathbb{R}} e^{-\pi x^2} dx \right)^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi (x^2+y^2)} dx\, dy$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r\, dr\, d\theta$$

$$\left(\text{Letting} \quad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right)$$

$$= \int_0^{2\pi} - \frac{e^{-\pi r^2}}{2\pi} \Bigg|_{r=0}^{\infty} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi}\, d\theta = 1.$$

Hence $\quad \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$, implying $\quad \hat{f}(0) = 1.$

Therefore $\quad \hat{f}(\xi)\, e^{\pi\xi^2} = \hat{f}(0) = 1 \Rightarrow \hat{f}(\xi) = e^{-\pi\xi^2}.$ ▨

§ 5.4 Inversion formula.

Thm 1. Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

To prove the theorem, we introduce the following definition.

Def. Let $f, g \in M(\mathbb{R})$. Set

$$f * g\,(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

Prop 2: Let $f, g \in M(\mathbb{R})$. Then

① $f * g = g * f$

② $f * g \in M(\mathbb{R})$

③ $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.

Pf: We only prove ②. The proofs of ① and ③ are Similar to that for convolutions of functions on the Circle.

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

$$= \int_{|y| \le \frac{|x|}{2}} + \int_{|y| \ge \frac{|x|}{2}} f(x-y)\, g(y)\, dy$$

$$= (\mathrm{I}) + (\mathrm{II})$$

$$|(\mathrm{I})| \le \int_{|y| \le \frac{|x|}{2}} |f(x-y)|\, |g(y)|\, dy$$

(notice that $|x-y| \ge \frac{|x|}{2}$)

$$\le \int_{|y| \le \frac{|x|}{2}} \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \cdot |g(y)|\, dy$$

$$\le \frac{A}{1+\frac{|x|^2}{4}} \int_{\mathbb{R}} |g(y)|\, dy$$

$$\le \frac{\tilde{A}}{1+|x|^2}$$

$$|(\mathrm{II})| \le \int_{|y| \ge \frac{|x|}{2}} |f(x-y)| \cdot |g(y)|\, dy$$

$$\le \int_{|y| \ge \frac{|x|}{2}} |f(x-y)| \cdot \frac{A}{1+\left(\frac{|x|}{2}\right)^2}\, dy$$

$$\leq \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \cdot \int_{\mathbb{R}} |f(x-y)|\, dy$$

$$\leq \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \int_{\mathbb{R}} |f(y)|\, dy$$

$$\leq \frac{\tilde{A}}{1+|x|^2}.$$

Hence $\displaystyle |f * g(x)| \leq \frac{A'}{1+|x|^2}$.

Also it is easy ~~to~~ show that $f * g$ is cts. ▨

<u>Def.</u> A family of $(K_t)_{t\in(a,b)} \subset M(\mathbb{R})$ is called a good kernel on $\mathbb{R}$ as $t \to t_0$, if

① $\displaystyle \int_{\mathbb{R}} K_t(x)\, dx = 1 \quad \forall\ t \in (a,b).$

② $\displaystyle \int_{\mathbb{R}} |K_t(x)|\, dx \leq M$ for some const $M > 0$, $\forall\ t \in (a,b)$

③ $\forall\ \delta > 0,$

$$\int_{|x|\geq\delta} |K_t(x)|\,dx \to 0 \quad \text{as } t\to t_0.$$

Thm (Convergence Thm):

If $(K_t)_{t\in(a,b)}$ is a good kernel on $\mathbb{R}$, as $t\to t_0$, and $f\in M(\mathbb{R})$, then

$$f*K_t(x) \rightrightarrows f(x) \quad \text{as } t\to t_0.$$

Pf. It is similar to the proof in the circle case. ▨

Thm (Multiplicative formula):

Let $f, g\in M(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}} \hat{f}(x)g(x)\,dx.$$

(Fubini Thm: Let $F(x,y)\in C(\mathbb{R}^2)$.

Suppose one of the 3 integrations are finite

① $\iint_{\mathbb{R}^2} |F(x,y)|\,dxdy$;

② $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)| \, dy \right) dx$;

③ $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)| \, dx \right) dy$;

Then

$$\iint_{\mathbb{R}^2} F(x,y) \, dx dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y) \, dy \right) dx$$
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y) \, dx \right) dy .$$

Now we prove the multiplicative formula:

Notice that

$$\int_{\mathbb{R}} f(x) \, \hat{g}(x) \, dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) \, g(y) \, e^{-2\pi i x y} \, dy \right) dx$$

by Fubini

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) g(y) e^{-2\pi i x y} \, dx \right) dy$$

(checking that $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x)| |g(y)| \, dy \right) dx < \infty$)

$$= \int_{\mathbb{R}} \hat{f}(y)\, g(y)\, dy$$

$\blacksquare$

Recall (Inversion Formula)

: If $f \in M(\mathbb{R})$ and $\hat{f} \in M(\mathbb{R})$, then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi .$$

Proof: We first prove that

$$f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\, d\xi .$$

Set for $\delta > 0$,

$$K_\delta(x) = e^{-\pi \delta x^2} .$$

Notice that $\widehat{K_\delta}(\xi) = \frac{1}{\sqrt{\delta}}\, e^{-\pi \xi^2/\delta} .$

Fact: $\left(\widehat{K_\delta}\right)_{\delta>0}$ is a good kernel as $\delta \to 0$.

Check:

① $$\int_{\mathbb{R}} \widehat{K_\delta}(x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta}\,dx$$

$$\underset{\text{Letting } y=\frac{x}{\sqrt{\delta}}}{=} \int_{\mathbb{R}} e^{-\pi x^2}\,dx$$

$$= 1$$

② $$\int_{\mathbb{R}} \left|\widehat{K_\delta}(x)\right| dx = 1$$

③ $\forall\ \gamma > 0$

$$\int_{|x|\geq\gamma} \left|\widehat{K_\delta}(x)\right| dx = \int_{|x|\geq\gamma} \frac{1}{\sqrt{\delta}} e^{-\pi \frac{x^2}{\delta}}\,dx$$

$$\underset{\text{Letting } y=\frac{x}{\sqrt{\delta}}}{=} \int_{|y|\geq\frac{\gamma}{\sqrt{\delta}}} e^{-\pi y^2}\,dy$$

$$\to 0 \quad \text{as} \quad \delta \to 0$$

Now by the convergence Thm,

$$f(0) = \lim_{\delta \to 0} f * \widehat{K_\delta}(0)$$

$$= \lim_{\delta \to 0} \int_{\mathbb{R}} f(x)\, \widehat{K_\delta}(-x)\, dx$$

$$= \lim_{\delta \to 0} \int_{\mathbb{R}} f(x)\, \widehat{K_\delta}(x)\, dx$$

$\left(\text{Using } \widehat{K_\delta}(x) = \widehat{K_\delta}(-x) \right)$

by Multiplicative formula

$$= \lim_{\delta \to 0} \int_{\mathbb{R}} \hat{f}(x)\, K_\delta(x)\, dx$$

Dominated convergence Thm

$$= \int_{\mathbb{R}} \lim_{\delta \to 0} \hat{f}(x)\, K_\delta(x)\, dx$$

$\left(K_\delta(x) = e^{-\pi \delta x^2} \right)$

$$= \int_{\mathbb{R}} \hat{f}(x)\, dx$$

This proves the inversion formula at $x = 0$.

Now for any $x_0 \in \mathbb{R}$, define

$$f_{x_0}(x) = f(x + x_0).$$

Then

$$f_{x_0}(0) = \int_{\mathbb{R}} \widehat{f_{x_0}}(\xi)\, d\xi$$

$$\text{LHS} = f(x_0)$$

$$\text{RHS} = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x_0}\, d\xi$$

So we obtain

$$f(x_0) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x_0}\, d\xi$$

$\blacksquare$.